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LYAPUNOV STABILITY WITH RESPECT TO GIVEN STATE FUNCTIONS*

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The general formulation of the problem of the stability of motion with respect to prescribed functions of the coordinates and velocities is due to Lyapunov /1/. A special case is that of partial stability, i.e., stability with respect to some of the variables /2-4/. In this paper some ideas of Lyapunov's first method are applied to the general problem of the stability of the equilibrium of reversible systems. The study is based on an examination of the trajectories asymptotic to an equilibrium position: if the equilibrium is stable with respect to a function Q, this function is stable on the asymptotic trajectories. Asymptotic solutions are sought using a special kind of series. As an application, a relativistic version of Earnshaw's theorem on the instability of the equilibrium of a charge in a stationary electric field is proved.

1. *Asymptotic solutions of reversible systems.* Let $x=(x_1,\ldots,x_n)$ be Lagrangian coordinates of a mechanical system.

$$T=\frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x)\,x_i^{\cdot}x_j^{\cdot}$$

its kinetic energy and $X(x)=(X_1,\ldots,X_n)$ a field of generalized forces. It will be assumed throughout that $a_{ij}, X_k\ (i,j,k=1,\ldots,n)$ are infinitely differentiable functions of x. If no additional constraints are imposed on the system, its motion is described by the Lagrange equations of the second kind:

$$\frac{d}{dt}\frac{\partial T}{\partial x_i^{\cdot}}-\frac{\partial T}{\partial x_i}=X_i\quad (i=1,\ldots,n) \tag{1.1}$$

Let us assume that $X(0)=0$. Then $x=0$ is an equilibrium position. We shall expand the functions $X_i(x)$ in series in terms of homogeneous forms in x: $X_i=X_i^{(m)}+X_i^{(m+1)}+\ldots$. As a rule, $m=1$. However, degenerate cases may occur in which $m\geqslant 2$. Define $X^{(m)}=(X_1^{(m)},\ldots,X_n^{(m)})$. Without loss of generality, we may assume that the matrix $\|a_{ij}(x)\|$ is the identity matrix at $x=0$.

Theorem 1. Suppose a vector e, $|e|=1$, exists, such that $X^{(m)}(e)=\varkappa e$, $\varkappa>0$. Then Eqs. (1.1) have a solution $x(t)$ for which the series

$$\sum_{k=1}^{\infty} x^{(k)}(t)e^{-\mu kt},\quad \mu=\varkappa^{1/2}>0,\quad \text{if}\quad m=1 \tag{1.2}$$

$$\sum_{k=1}^{\infty}\frac{x^{(k)}(\ln t)}{t^{k\mu}},\quad \mu=\frac{2}{(m-1)}>0,\quad \text{if}\quad m>1 \tag{1.3}$$

is an asymptotic expansion as $t\to+\infty$.

In formulae (1.2) and (1.3) $x^{(k)}(z)$ are certain polynomials in z, with $x^{(1)}=\lambda e$, $\lambda=\text{const}>0$. In particular, $x(t)\to 0$ as $t\to+\infty$.

The idea of the proof is as follows. One first looks for series (1.2), (1.3) which formally satisfy Eqs. (1.1). The series coefficients $x^{(k)}$ are determined successively by

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induction (see /5/). The series (1.2), (1.3) may be divergent. However, it was shown in /6/ that if that is the case Eqs.(1.1) nevertheless have a solution $x(t)$, for which one of the series is an asymptotic representation. For example, for the series (1.3) this means that

$$x(t)-\sum_{k=1}^{N}\frac{x^{(k)}(\ln t)}{t^{k\mu}}=o\left(\frac{\ln^{j}t}{t^{N\mu}}\right)$$

as $t\to+\infty$, where t is the degree of the vector polynomial $x^{(N)}$. Series of the form (1.2) were first used by Lyapunov in his treatment of the stability of motion /1/.

Remark. Let $x=0$ be an isolated zero of the homogeneous vector field $X^{(m)}(x)$. We know from topology that for odd n there is always a vector e, $|e|=1$, such that $X^{(m)}(e)=\varkappa e$. However, the factor $\varkappa$ may be negative.

Let $Q(x^{\cdot},x)$ be a smooth function in the phase space; we shall assume that $Q(0,0)=0$. We will consider the question of whether the equilibrium $x=0$ is stable with respect to Q. Let $x(t)$ be the asymptotic solution of system (1.1) whose existence is guaranteed by Theorem 1: $x(t)\to 0$ as $t\to+\infty$. By reversibility, Eqs.(1.1) have a solution $x'=x(-t)$ which tends to zero as $t\to-\infty$. Consider the function of time defined by

$$q'(t)=Q(x'^{\cdot}(t),x'(t))$$

If $q'(t)\to 0$ as $t\to-\infty$ and $q'(t)\not\equiv 0$, then the equilibrium position $x=0$ is clearly unstable with respect to Q. This condition is equivalently expressed as

$$q(t)=Q(-x^{\cdot}(t),\quad x(t))\to 0$$

as $t\to+\infty$. Substituting the series (1.2) (or (1.3)) into the Maclaurin expansion of $Q(-x^{\cdot},x)$, we obtain the series

$$\sum_{k=1}^{\infty}q^{(k)}(t)\,e^{-\mu kt}\left(\sum_{k=1}^{\infty}\frac{q^{(k)}(\ln t)}{t^{k\mu}}\right)\tag{1.4}$$

Here $q^{(k)}(z)$ are certain polynomials in z with constant coefficients. The series (1.4) is clearly an asymptotic representation of $q(t)$ as $t\to+\infty$. We have thus proved the following theorem.

Theorem 2. If at least one coefficient of the formal series (1.4) does not vanish, the equilibrium position $x=0$ is unstable with respect to Q.

For example, suppose that the forces have a potential and that the Maclaurin expansion of the potential energy begins with a non-trivial homogeneous form V_{m+1} of degree $m+1$. Then obviously $X^{(m)}=-\partial V_{m+1}/\partial x$. It can be shown that if V_{m+1} is not a minimum at $x=0$, the equilibrium position is unstable, e.g., with respect to the Lagrange function. The vector e of Theorem 1 will be a minimum point of the function V_{m+1} on the unit sphere $|x|=1$ (see /7/).

Theorems 1 and 2 can be generalized to non-holonomic systems with stationary homogeneous constraints

$$\sum_{i=1}^{n}b_{ij}(x)\,x_i^{\cdot}=0,\quad j=1,\ldots,m<n\tag{1.5}$$

Eqs.(1.1) are replaced by the following, more general equations:

$$\frac{d}{dt}\frac{\partial T}{\partial x_i^{\cdot}}-\frac{\partial T}{\partial x_i}=X_i+\sum_{j=1}^{m}\lambda_j b_{ij}\tag{1.6}$$

Eqs.(1.5) and (1.6) must be treated simultaneously. Let $X_*^{(m)}$ be the orthogonal projection of the homogeneous field $X^{(m)}$ onto the plane Π defined by the equations

$$\sum_{j=1}^{n}b_{ij}x_i=0,\quad j=1,\ldots,m$$

Theorem 3. Assume that $X_*^{(m)}(e)=\varkappa e$, where e is the unit vector in Π, $\varkappa=\text{const}>0$. Then the system of Eqs.(1.5), (1.6) has solutions with asymptotic expansions (1.2), (1.3). In particular, the equilibrium position $x=0$ is unstable.

This theorem extends the results of /8/ to non-potential fields.

2. ***Relativistic version of Earnshaw's theorem.*** The relativistic equation of motion of a charged particle is

$$[\, mx^{\cdot} (1 - x^{\cdot 2}/c^2)^{-1/2}]^{\cdot} = F, \quad x \in \mathbf{R}^3 \tag{2.1}$$

where $F = q(E + x^{\cdot} \times H)$ is the Lorentz force, m is the mass, q is the charge of the particle, c is the speed of light and $E\,(H)$ is the electric (magnetic) field strength. Eq.(2.1) may be rewritten as a "Newton equation" (see /9/):

$$mx^{\cdot\cdot} = [F - (x^{\cdot}/c^2)\,(F, x^{\cdot})]\,(1 - x^{\cdot 2}/c^2)^{1/2} \tag{2.2}$$

Suppose $H \equiv 0$ and the field E does not depend explicitly on time. Then Eq.(2.2) is invertible.

Theorem 4. The equilibrium of a charge in a stationary electric field is always unstable.

This proposition extends the fundamental theorem of Earnshaw /10/ to the relativistic case. A stationary electric field is irrotational, and its potential is in fact a harmonic function. Any homogeneous form of the Maclaurin series of a harmonic function is again a harmonic function. In particular, by the Mean-Value Theorem, the first non-trivial form does not have a local minimum at an equilibrium position. Using the method of Sect.1 one can prove the existence of asymptotic solutions in the form of series (1.2), (1.3) (see /11/). Since Eqs.(2.2) are reversible, the equilibrium is unstable.

Of course, linearization of (2.2) leads to the usual non-relativistic equation, and instablity can be proved by means of the classical Earnshaw Theorem. In degenerate cases, however, conclusions relating to stability can no longer be based on an analysis of the linearized equations.

It is clear that the equilibrium of a charge is unstable with respect to the components of the electric field and its potential.

3. ***Some generalizations.*** Consider an autonomous system of equations

$$x^{\cdot} = v(x), \quad x \in \mathbf{R}^n \tag{3.1}$$

Let $v(0) = 0$. Then $x = 0$ is an equilibrium position of system (3.1). We wish to investigate its stability with respect to a smooth function $Q(x)$, where $Q(0) = 0$.

Time reversal applied to (3.1) produces the following new system of equations;

$$x^{\cdot} = -v(x). \tag{3.2}$$

Lemma. Let system (3.2) have a solution $x(t)$ such that

1) $x(t) \to 0$ as $t \to +\infty$, 2) $q(t) = Q(x(t)) \not\equiv 0$.

Then $x = 0$ is an unstable equilibrium position of system (3.1) with respect to Q.

Asymptotic solutions of Eqs.(3.2) may be sought as series of a certain type. Expanding the components of the vector field v in Maclaurin series, we write system (3.2) in the form of the equations

$$x^{\cdot} = Ax + \ldots \tag{3.3}$$

Let e be an eigenvector of the matrix A with a real eigenvalue $-\mu < 0$. Then Eqs.(3.3) have a particular solution which is a series (1.2) with $x^{(1)} = e$. Inserting this series into the Maclaurin expansion of Q, we again obtain a series in powers of $\exp(-\mu t)$ whose coefficients are polynomials in t. If at least one coefficient of this series does not vanish, the equilibrium of system (3.1) is unstable with respect to Q.

This observation can be generalized. The asymptotic solutions can be expanded in multiple series in terms of exponential functions, which are also suitable for use when the eigenvalues of A are complex /1/. A necessary condition for the equilibrium position $x = 0$ of system (3.1) to be stable with respect to Q is that this function be constant on the central manifold of system (3.2). This property can be verified constructively by using an iterative method to construct the Lyapunov series.

In degenerate cases the asymptotic solutions may be sought as series (1.3).

As an example, consider the critical case of one zero root /1/. In a typical situation Eqs.(3.2) can be reduced to the form

$$z^{\cdot} = az^2 + f(z, y) + O(|x|^3), \quad y^{\cdot} = By + O(|x|^2) \tag{3.4}$$
$$z \in \mathbf{R}, \quad y \in \mathbf{R}^{n-1}; \quad x = (z, y)$$

Here $a = \text{const} \neq 0$, B is a non-singular matrix and f is a quadratic polynomial not containing z^2.

Theorem 5. Under the above assumptions, Eqs.(3.4) have an asymptotic solution given by the series

$$z = \sum_{k=1}^{\infty} \frac{z^{(k)}(\ln t)}{t^k}, \quad y = \sum_{m=2}^{\infty} \frac{y^{(m)}(\ln t)}{t^m} \tag{3.5}$$

where $z^{(k)}(\cdot), y^{(m)}(\cdot)$ are polynomials, $z^{(1)} = -1/a = \text{const}$.

Proof. Inserting the series (3.5) into Eqs.(3.4) and equating coefficients of like powers $1/t^{k+1}$, we obtain an infinite chain of equations for the successive determination of the coefficients $z^{(k)}$ and $y^{(k+1)}$:

$$z^{(k)\prime} = (k-2)\, z^{(k)} + g_k, \quad By^{(k+1)} = G_{k+1}, \quad k = 2, 3, \ldots \tag{3.6}$$

Here g_k, G_{k+1} are certain known polynomials in $\ln t$ and the prime indicates differentiation with respect to $\ln t$. By (3.5), $z^{(k)}$ and $y^{(k+1)}$ are found as polynomials in $\ln t$. The series (3.5) are generally divergent, but Eqs.(3.4) still have solutions for which the series (3.5) are asymptotic expansions /6/.

The following example shows that the series (3.5) may diverge even in the analytic case. The system

$$z^{\cdot} = z^2, \quad y^{\cdot} = y - z^2 \tag{3.7}$$

is an instance of (3.4), with the formal solution

$$z = -\frac{1}{t}, \quad y = \sum_{k=2}^{\infty} \frac{(-1)^k (k-1)!}{t^k} \tag{3.8}$$

The series for $y(t)$ is divergent for all $t > 0$, but system (3.7) has the asymptotic solution

$$z(t) = -\frac{1}{t}, \quad y(t) = e^t \int_t^{+\infty} \frac{e^{-u}}{u^2}\, du \tag{3.9}$$

Integrating successively by parts, we obtain the asymptotic series (3.8). The passage from (3.8) to (3.9) may be interpreted as summation of a divergent series /12/.

The existence of asymptotic solutions in the form of series (3.5) enables one to establish simple sufficient conditions for the equilibrium position $x = 0$ of system (3.4) to be unstable with respect to prescribed smooth functions of $x = (y, z)$.

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